Definition 2.3. Let Y be a random variable mapping the probability space $(\Omega, \mathcal{F}, P)$ into the measurable space $(\mathcal{Y}, \mathcal{B})$ such that $\mathcal{E}|Y| < \infty$. Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$. The *conditional expectation* of Y given $\mathcal{F}_0$, denoted $\mathcal{E}(Y|\mathcal{F}_0)$, is an $\mathcal{F}_0$-measurable function $h^*(\omega)$ that satisfies

$$\int_F Y(\omega)\, dP(\omega) = \int_F h^*(\omega)\, dP(\omega)$$

for every $F \in \mathcal{F}_0$. Equivalently, $\mathcal{E}(Y|\mathcal{F}_0)$ is a random variable mapping $(\Omega, \mathcal{F}, P)$ into $(\mathcal{Y}, \mathcal{B})$ that is measurable $\mathcal{F}_0/\mathcal{B}$ and that satisfies $\mathcal{E}(I_F Y) = \mathcal{E}[I_F \mathcal{E}(Y|\mathcal{F}_0)]$ for every $F \in \mathcal{F}_0$.

Also, note that $\mathcal{E}(Y|\mathcal{F}_0)$ and $\mathcal{E}(Y|X)$ are functions. Writing $\mathcal{E}(Y|\mathcal{F}_0)(\omega)$, $\mathcal{E}(Y|X)(x)$, or $\mathcal{E}(Y|X)[X(\omega)]$ calls attention to their arguments. If $\mathcal{F}_0 = X^{-1}(\mathcal{A})$, then the relation among them is

$$\mathcal{E}(Y|\mathcal{F}_0)(\omega) = \mathcal{E}(Y|X)(x)\big|_{x=X(\omega)} = \mathcal{E}(Y|X)[X(\omega)].$$